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Numerical Solution of the Compressible Boundary-Layer Equations Using the Finite Element Method

E. Hytopoulos,* J. A. Schetz,† and M. Gunzburger‡
Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061

Abstract

THIS paper concerns the application of the finite element method (FEM) to the boundary-layer equations governing both incompressible (with heat transfer) and compressible fluids in two dimensions. Numerical solutions for this set of equations have traditionally been obtained using the finite difference method (FDM), where the derivatives are approximated by difference quotients resulting in an algebraic representation of the partial differential equations.

In FEM, the flow region is subdivided into a number of small regions called finite elements, and the dependent variables are interpolated within each element by functions of compatible order. Using these approximations in the equations, we introduce errors. The Galerkin method seeks to reduce these errors to zero in a weighted sense; thus the partial differential equations describing the problem in the region as a whole are replaced by algebraic equations in each element.

In this paper, a Galerkin method is considered for approximating solutions of the compressible boundary-layer equations. The method has recently been applied to the solution of the laminar and turbulent incompressible case with excellent results.[1] The parabolic nature of the problem allows the solution of the algebraic system for one column of elements at a time. This solves one of the biggest problems in using FEM, the simultaneous solution of a large number of equations, which is generally associated with high computational cost.

The FEM was chosen for several reasons. First, this method has been successfully applied to the full Navier-Stokes equations. Second, the purely "local" approximations of the phenomena effectively free the analyst from difficulties associated with irregular geometries, multiconnected domains, and mixed boundary conditions when the problem requires. Third, applications are firmly rooted in the physics of the problem, and for a given accuracy, some studies indicate that the resulting equations are better conditioned than those obtained by finite difference approximations.[2] Fourth, because the FEM is based on a local analytic solution and is formulated via an integral relationship, other studies contend that finite element solutions are more accurate than finite difference solutions for the same number of unknowns.[3]

With the boundary layer assumptions, the continuity equation cannot be treated as a constraint as with Navier-Stokes formulations. It is now an ordinary differential equation, numerical integration of which yields the required solution at the nodes. Also, the pressure is not an unknown.

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*Graduate Assistant, Aerospace and Ocean Engineering Department. Student Member AIAA.

†W. Martin Johnson Professor and Head, Aerospace and Ocean Engineering Department. Fellow AIAA.

‡Professor, Department of Mathematics.

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Analysis

If we denote by $A(x, y)$ the approximation to the dependent variable over an element, a_i the values of the variable at node i of the element, and ϕ_i the interpolation functions over an element, then the finite element approximation over the element can be written in the following form:

$$A(x, y) = \sum_{i=1}^{N} a_i \phi_i \tag{1}$$

where $A(x, y)$ represents $[\rho U](x, y)$, $[\rho V](x, y)$, or $T(x, y)$. Introducing these approximations in the equations, one gets

$$f_j(\phi, a_i) = R_j \neq 0 \tag{2}$$

The Galerkin method of weighted residuals seeks to reduce these errors to zero in a weighted sense by making the residual orthogonal to the interpolation functions of each element. For the compressible flow case, application of the Galerkin method to the momentum and energy equations leads to the following matrix problem for each element:

$$[K][\rho u] = [F] \tag{3}$$

where the stiffness matrix K and the force vector f over each element are

$$K_{ij} = \int_{\Omega_e} \left[\bar{u}\phi_i \frac{\partial \phi_j}{\partial x} + \bar{v}\phi_i \frac{\partial \phi_j}{\partial y} + \phi_i \phi_j \left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right) + \frac{\mu}{\rho} \frac{\partial \phi_i}{\partial y} \frac{\partial \phi_j}{\partial y} \right] \mathrm{d}x\, \mathrm{d}y \tag{4}$$

$$F_i = \int_{\Omega_e} \frac{\mu}{\rho} \frac{\partial \bar{\rho}}{\partial y} \bar{u} \frac{\partial \phi_i}{\partial y} \mathrm{d}x\, \mathrm{d}y + \int_{\Gamma_e} \mu \frac{\partial u}{\partial y} \phi_i \eta_y \, \mathrm{d}s - \int_{\Omega_e} \frac{\partial P}{\partial x} \phi_i \, \mathrm{d}x\, \mathrm{d}y \tag{5}$$

for the momentum equation and

$$[K][T] = [F] \tag{6}$$

where

$$K_{ij} = c_P \left\{ \int_{\Omega_e} \left[\overline{(\rho u)} \frac{\partial \phi_j}{\partial x} \phi_i + \overline{(\rho v)} \frac{\partial \phi_j}{\partial y} \phi_i + \frac{\mu}{Pr} \frac{\partial \phi_j}{\partial y} \frac{\partial \phi_i}{\partial y} \right] \mathrm{d}x\, \mathrm{d}y \right\} \tag{7}$$

$$F_i = \int_{\Omega_e} \left[\bar{u} \frac{\partial P}{\partial x} \phi_i + \mu \left(\frac{\partial \bar{u}}{\partial y} \right)^2 \phi_i \right] \mathrm{d}x\, \mathrm{d}y + \int_{\Gamma_e} \kappa \frac{\partial T}{\partial y} \phi_i \eta_y \, \mathrm{d}s \tag{8}$$

for the energy equation. The divergence theorem has been applied to the viscous and heat flux terms to reduce the order of differentiation. The nonlinearity of the convective terms requires an iterative solution of the system. The terms $\overline{(\rho u)}$, $\overline{(\rho v)}$ are evaluated using a simple iterative updating procedure. At the nth iteration, the values obtained during the $(n - 1)$ iteration are used for evaluating the coefficients, and

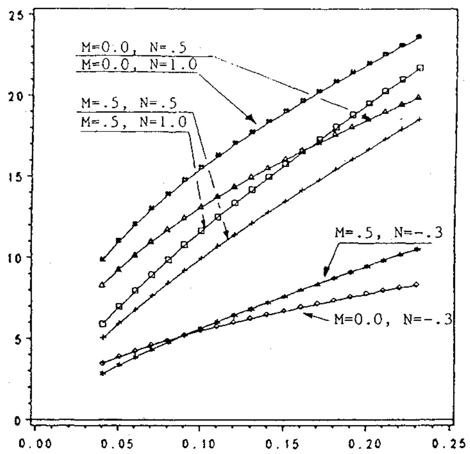

Fig. 1 Nusselt number for different pressure and wall temperature gradients; analytical (—) vs FEM (symbols) solution for laminar, incompressible flow with heat transfer.

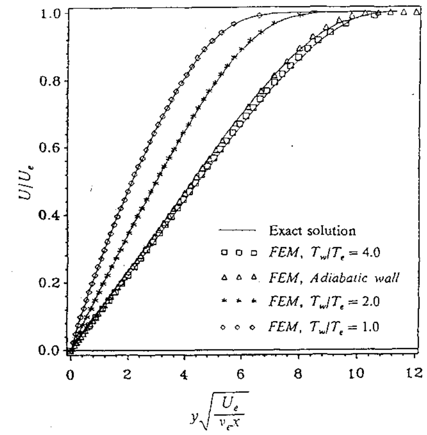

Fig. 2 Compressible flow for Mach = 4.0; analytical vs FEM solution. Nondimensional velocity profile for T_w/T_e = 1.0, 2.0, and 4.0 and adiabatic wall.

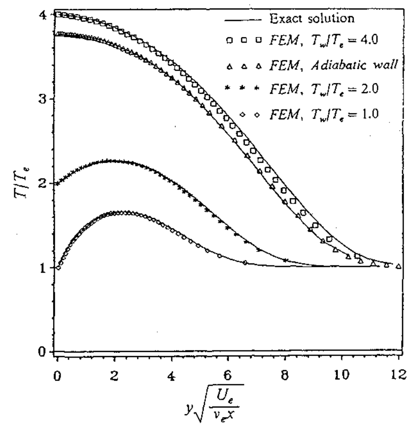

Fig. 3 Compressible flow for Mach = 4.0; analytical vs FEM solution. Nondimensional temperature profile for T_w/T_e = 1.0, 2.0, and 4.0 and adiabatic wall.

the system is solved for the new values of the dependent variables. The continuity equation should be integrated along the vertical direction, starting from the known value for the ρv component on the lower boundary after the x-momentum equation is solved for the ρu nodal values.

In this paper, only results based on a four-node quadrilateral element with bilinear interpolation of the dependent variables are presented.

The numerical evaluation of the integrals is performed using a quadrature rule compatible with the order of the elements used. For the linear element, a 2×2 quadrature integrates the expressions exactly. The K matrix for each element has dimensions 4×4, and the F vector has four elements. The values of the nodal unknowns on the upstream face of the element are known from the solution of the previous column; thus they can be applied as boundary conditions at the element level. Modification of the element matrix leads to a 2×2 matrix. Solution for the column of elements requires the assembly of the element stiffness matrix into a global matrix and assembly of the element force vector into a global force vector. The assembly procedure leads to a tridiagonal system of equations for the nodes on the downstream side of the column in the case of the linear elements. The Thomas algorithm is used for the solution of the tridiagonal system.

For all of the cases, a geometric progression with ratios between 1.01 and 1.08 has been used to create a variable spacing of the nodes in the y direction. The number of elements used varied between 30 and 45. This is a rather fine grid, but we were looking to test the ultimate accuracy of the basic method. A uniform step size in the x direction was employed with Δx = 10 to 20 times the minimum Δy.

Results

To validate the approach, problems corresponding to both incompressible (with heat transfer) and compressible flow cases have been solved.

The first case tested was low-speed flow with constant properties and heat transfer. In this case, the energy equation is decoupled from the momentum equation and is solved after the solution for the velocity field is obtained.

The similar solutions for variable pressure[4] and wall temperature[5] gradient were used for comparison. Figure 1 presents some results, where M and N are the exponents describing the variation of the external velocity and wall temperature as a function of the x location, i.e., $U_e = U_1 x^M$ and $(T_w - T_e) = Ax^N$.

The solutions obtained by Van Driest[6] using the Crocco method are used for testing the performance of the new FEM for compressible flows. Typical results are presented in terms of nondimensionalized velocity and temperature profiles. Figures 2 and 3 show the temperature and velocity profiles for T_w/T_e equal to 1.0, 2.0, and 4.0 and for the adiabatic wall case at Mach 4.0.

The grid independence was checked, varying the *Re* for the same grid. Grid independent solutions were obtained for all the cases.

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